

# Entropy, topology of two-dimensional extreme black holes

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## Abstract

Through direct thermodynamic calculations we have shown that different classical entropies of two-dimensional extreme black holes appear due to two different treatments, namely Hawking's treatment and Zaslavskii's treatment. Geometrical and topological properties corresponding to these different treatments are investigated. Quantum entropies of the scalar fields on the backgrounds of these black holes concerning different treatments are also exhibited. Different results of entropy and geometry lead us to argue that there are two kinds of extreme black holes in the nature. Explanation of black hole phase transition has also been given from the quantum point of view.

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# 1 Introduction

The traditional Bekenstein-Hawking entropy of black hole, which is known to be proportional to the area  $A$  of the horizon, is believed to be appropriate to all kinds of black holes, including the extreme black hole (EBH). However, recently, based upon the study of topological properties, it has been argued that the Bekenstein-Hawking formula of the entropy is not valid for the EBH [1,2]. The entropy of four-dimensional (4D) extreme Reissner-Nordstrom (RN) black hole is zero regardless of its nonvanishing horizon area. Further study of the topology [2-4] shew that the Euler characteristic of such kind of EBH is zero, profoundly different from that of the nonextreme black hole (NEBH). From the relationship between the topology and the entropy obtained in ref.[4], we see that this extreme topology naturally results in the zero entropy.

But these results meet some challenges. By means of the recalculation of the proper distance between the horizon and any fixed point, Zaslavskii [5] argued that a 4D RN black hole in a finite size cavity can approach the extreme state as closely as one likes but its entropy as well as its temperature on the cavity are not zero. Entropy is still proportional to the area. Zaslavskii's result has also been supported by string theorists got by counting string states [6]. The geometry of EBH obtained in the approach of Zaslavskii has been studied in [7,8] and it was claimed that its topology is of the nonextreme sector. Meanwhile the string theorists' results were interpreted by summing over the topology [9], however this viewpoint has been refuted by Zaslavskii [10,11]. These different results indicate that there is a clash for the understanding of the topology as well as the intrinsic thermodynamical properties of EBHs. Comparing [1,2] and [5,7,8], this clash seems come from two different treatments: one refers to Hawking's treatment by starting with the original EBH [1,2] and the other Zaslavskii's treatment by first taking the boundary limit and then the extreme limit to get the EBH from nonextreme counterpart [5,7,8]. Recently we have studied the geometry and intrinsic thermodynamics of extreme Kerr black hole by using these two treatments and found that these different treatments approach to two different topological objects and lead to drastically different intrinsic thermodynamical properties [12]. Of course, all these results obtained are limited in the classical treatment of 4D black holes.

The motivation of the present paper is to extend 4D classical studies to two-dimensional (2D) black holes. We hope that the mathematical simplicity in 2D black holes can help us to understand the problem clearer and deeper. The results on 2D charged dilaton black hole topology and thermodynamics were only announced and briefly summarized in [13,14]. To make the study general, we will study two kinds of 2D black holes, the 2D charged dilaton black hole [15,16] as well as the 2D Lowe-Strominger black

hole [17], by using two treatments mentioned above in detail. We will prove that these two treatments result in two different thermal results: Bekenstein-Hawking entropy and zero entropy for EBHs. Besides we will investigate the geometry and topology of 2D EBHs in detail and directly relate two treatments to topological properties of EBHs. We will clearly exhibit Euler characteristic values for 2D EBHs, especially for EBHs obtained from the nonextreme counterpart by Zaslavskii's treatment. Different Euler characteristics are directly derived from different treatments, rather than by introducing some other conditions, such as the inner boundary condition as done in 4D cases [3].

The other objective of the present paper is to study this problem quantum mechanically. As early pointed out by t'Hooft[18], the fields propagating in the region just outside the horizon give the main contribution to the black hole entropy. Many methods, for example, the brick wall model[18,19], Pauli-Villars regular theory[20] etc., have been suggested to study the quantum effects of entropy under WKB approximation or one-loop approximation. Suppose the black hole is enveloped by a scalar field, and the whole system, the hole and the scalar field, are filling in a cavity. Adopt the viewpoint that the entropy arises from entanglement[21-23], it is of interest to study the quantum effects of these two different treatments on the entropy of the scalar field on the EBHs' backgrounds under WKB approximation, in particular, to investigate whether these two different treatments will offer two different values of entropy. In Sec.IV we will prove that the entropy of the scalar field depends on two different treatments as well. Some physical understanding concerning these results will also be given.

The organization of the paper is as the following: In Sec.II, the classical entropy of two kinds of 2D EBHs are derived by using two different treatments. And in Sec.III, the geometry and topology of these EBHs are investigated. The Euler characteristics are clearly exhibited. Sec.IV is devoted to the discussion of the entropy of the scalar field on the EBH background. The conclusions and discussions will be presented in the last section.

## 2 Classical entropy

We first study the 2D charged dilaton black hole(CDBH)[15,16]. The action is

$$I = - \int_M \sqrt{g} e^{-2\phi} [R + 4(\nabla\phi)^2 + \lambda^2 - \frac{1}{2}F^2] - 2 \int_{\partial M} e^{-2\phi} K \quad (1)$$

which has a black hole solution with the metric

$$ds^2 = -g(r)dt^2 + g^{-1}(r)dr^2 \quad (2)$$

$$g(r) = 1 - 2me^{-\lambda r} + q^2 e^{-2\lambda r} \quad (3)$$

$$e^{-2\phi} = e^{-2\phi_0} e^{\lambda r}, \quad A_0 = \sqrt{2}q e^{-\lambda r} \quad (4)$$

where  $m$  and  $q$  are the mass and electric charge of the black hole respectively. The horizons are located at  $r_{\pm} = (1/\lambda) \ln(m \pm \sqrt{m^2 - q^2})$ .

Using the finite-space formulation of black hole thermodynamics, employing the grand canonical emsemble and putting the black hole into a cavity as usual[5,24,25], we calculate the free energy and entropy of the CDBH. To simplify our calculations, we introduce a coordinate transformation

$$r = \frac{1}{\lambda} \ln[m + \frac{1}{2}e^{\lambda(\rho+\rho_0^*)} + \frac{m^2 - q^2}{2}e^{-\lambda(\rho+\rho_0^*)}] \quad (5)$$

where  $\rho_0^*$  is an integral constant, and rewrite Eq.(2) to a particular gauge

$$ds^2 = -g_{00}(\rho)dt^2 + d\rho^2 \quad (6)$$

The Euclidean action takes the form

$$I = - \int_{\partial M} \sqrt{\frac{1}{g_{11}}} e^{-2\phi} \left( \frac{1}{2} \frac{\partial_1 g_{00}}{g_{00}} - 2\partial_1 \phi \right) \quad (7)$$

The dilaton charge is found to be

$$D = e^{-2\phi_0} \left( m + \frac{1}{2}e^x + \frac{m^2 - q^2}{2}e^{-x} \right) \quad (8)$$

$$x = \lambda(\rho + \rho_0^*) \quad (9)$$

The free energy,  $F = I/\beta$ , where  $\beta$  is the proper periodicity of Euclideanized time at a fixed value of the special coordinate and has the form  $\beta = 1/T_w = \sqrt{g_{00}}/T_c$ ,  $T_c$  is the inverse periodicity of the Euclidean time at the horizon

$$T_c = \frac{\lambda \sqrt{m^2 - q^2}}{2\pi(m + \sqrt{m^2 - q^2})} \quad (10)$$

Using the formula of entropy

$$S = -(\partial F / \partial T_w)_D \quad (11)$$

we obtain

$$S = \frac{2\pi e^{-2\phi} [m + \frac{e^x}{2} + \frac{(m^2 - q^2)e^{-x}}{2}] [1 + (m^2 - q^2)e^{-2x}] \sqrt{m^2 - q^2} (m + \sqrt{m^2 - q^2})}{(m^2 - q^2) + m [\frac{e^x}{2} + \frac{(m^2 - q^2)e^{-x}}{2}]} \quad (12)$$

Taking the boundary limit  $x \rightarrow x_+ = \lambda(\rho_+ + \rho_0^*) = \ln \sqrt{m^2 - q^2}$  in Eq.(12) to get the entropy of the hole, we find

$$S = 4\pi e^{-2\phi_0} (m + \sqrt{m^2 - q^2}) \quad (13)$$

This is just the result given by Nappi and Pasquinucci[15] for the non-extreme CDBH, which confirms that our treatment above is right.

We are now in a position to extend the above calculations to EBH. We are facing two limits, namely, the boundary limit  $x \rightarrow x_+$  and the extreme limit  $q \rightarrow m$ . We can take the limits in different orders: (A) by first taking the boundary limit  $x \rightarrow x_+$ , and then the extreme limit  $q \rightarrow m$  as the treatment adopted in [5,7,8]; and (B) by first taking the extreme limit  $q \rightarrow m$  and then the boundary limit  $x \rightarrow x_+$ , which corresponds to the treatment of Hawking et al. [1,2] by starting with the original EBH. To do our limits procedures mathematically, we may take  $x = x_+ + \epsilon, \epsilon \rightarrow 0^+$  and  $m = q + \eta, \eta \rightarrow 0^+$ , where  $\epsilon$  and  $\eta$  are infinitesimal quantities with different orders of magnitude, and substitute them into Eq.(12). It can easily be shown that in treatment (A)

$$S_{CL}(A) = 4\pi m e^{-2\phi_0} \quad (14)$$

which is just the Bekenstein-Hawking entropy. However, in treatment (B),

$$S_{CL}(B) = 0 \quad (15)$$

which is just the result given by refs.[1,2].

These peculiar results can also be found in 2D Lowe-Strominger black hole. This 2D black hole is obtained in [17] by introducing gauge fields through the dimensional compactification of three-dimensional string effective action. The 2D action in this case has the form

$$I = - \int_M \sqrt{g} e^{-2\phi} [R + 2\lambda^2 - \frac{1}{4} e^{-4\phi} F^2] - 2 \int_{\partial M} e^{-2\phi} K \quad (16)$$

where  $\phi$  is a scalar field coming from the compactification and plays the role of dilaton for the 2D action. This action possesses the black hole solution

$$ds^2 = -(\lambda^2 r^2 - m + \frac{J^2}{4r^2}) dt^2 + \frac{1}{\lambda^2 r^2 - m + \frac{J^2}{4r^2}} dr^2 \quad (17)$$

$$A_0 = -\frac{J}{2r^2} \quad (18)$$

$$e^{-2\phi} = r \quad (19)$$

The parameter  $J$  in this solution gives “charge” to this black hole[25]. The horizons of this black hole locate at

$$r_{\pm} = \frac{1}{\lambda} \left\{ \frac{m}{2} [1 \pm (1 - (\frac{\lambda J}{m})^2)^{1/2}] \right\}^{1/2} \quad (20)$$

where  $r_+$  is the event horizon and  $r_-$  the inner cauchy horizon. In the extreme limit  $\lambda J \rightarrow m$ ,  $r_+$  and  $r_-$  degenerate.

Using the finite space formulation of black hole thermodynamics, employing the grand canonical ensemble and putting the hole into a cavity as usual[5,24], we calculate the free energy and the entropy of the hole. As done in 2D CDBH, we introduce a coordinate transformation to simplify the calculation

$$r^2 = \frac{1}{2\lambda^2} (m + \frac{1}{2} e^{2\lambda(\rho+\rho_0)} + \frac{m^2 - \lambda^2 J^2}{2} e^{-2\lambda(\rho+\rho_0)}) \quad (21)$$

where  $\rho_0$  is an integral constant, and rewrite Eq(17) to a particular gauge Eq(6). After transformation, the event horizon locates at

$$\rho_+ = \frac{1}{2\lambda} \ln \sqrt{m^2 - \lambda^2 J^2} - \rho_0 \quad (22)$$

The Euclidean action can be evaluated as

$$I = - \int_{\partial M} [n^a F_{ab} A^b e^{-6\phi} + 2K e^{-2\phi}] \quad (23)$$

The free energy can be obtained by the evaluation of (23). Employing the constant shift in the gauge potential  $A_a \rightarrow A_a + \text{constant}$ , in the equations of motion to avoid divergence in the gauge potential at the horizon as done in [25,26], we have the free energy

$$F = -2\lambda D \frac{e^{2x} + (m^2 - \lambda^2 J^2) e^{-2x} + 2\sqrt{m^2 - \lambda^2 J^2}}{e^{2x} - (m^2 - \lambda^2 J^2) e^{-2x}} \quad (24)$$

where  $x = \lambda(\rho + \rho_0)$ ,  $D$  is the dilaton charge given by

$$D = [\frac{1}{2\lambda^2} (m + \frac{1}{2} e^{2x} + \frac{m^2 - \lambda^2 J^2}{2} e^{-2x})]^{1/2} \quad (25)$$

Using Eq.(11), where  $T_w$  is the inverse periodicity of the Euclideanized time at a fixed value of the special coordinate.  $T_c$  is the inverse periodicity at the horizon, reads

$$T_c = \frac{\sqrt{2}\lambda\sqrt{m^2 - \lambda^2 J^2}}{2\pi(m + \sqrt{m^2 - \lambda^2 J^2})^{1/2}} \quad (26)$$

We find the entropy as

$$S = \frac{4\pi\sqrt{m^2 - \lambda^2 J^2} (m + \frac{e^{2x}}{2} + \frac{m^2 - \lambda^2 J^2}{2} e^{-2x}) e^{-2x} (\frac{e^{2x}}{2} + \frac{m^2 - \lambda^2 J^2}{2} e^{-2x} + \sqrt{m^2 - \lambda^2 J^2})}{\sqrt{2}\lambda [(m + \frac{e^{2x}}{2} + \frac{m^2 - \lambda^2 J^2}{2} e^{-2x})^2 - \lambda^2 J^2]} \times (m + \sqrt{m^2 - \lambda^2 J^2})^{1/2} \quad (27)$$

Taking the boundary limit

$$x \rightarrow x_+ = \lambda(\rho_+ + \rho_0) = \frac{1}{2} \ln \sqrt{m^2 - \lambda^2 J^2}$$

in Eq(27), one has

$$S = \frac{4\pi}{\sqrt{2}\lambda} (m + \sqrt{m^2 - \lambda^2 J^2})^{1/2} \quad (28)$$

This is just the result given in [25] for the NEBH.

As in the 2D CDBH we are facing two limits to extend the above calculation to EBH, namely, the boundary limit  $x \rightarrow x_+$  and the extreme limit  $\lambda J \rightarrow m$ . And again there are two treatments: (A) Zaslavskii's treatment by first taking the boundary limit  $x \rightarrow x_+$ , and then the extreme limit  $\lambda J \rightarrow m$ ; and (B) Hawking et al.'s treatment by first taking the extreme limit  $\lambda J \rightarrow m$  and then the boundary limit  $x \rightarrow x_+$ . It is easy to find for treatment (A)

$$S_{CL}(A) = \frac{2\sqrt{2}\pi}{\lambda} m^{1/2} \quad (29)$$

This is just the Bekenstein-Hawking entropy. It depends on the mass  $m$  only.

However, in treatment (B), we find

$$S_{CL}(B) = 0 \quad (30)$$

This is just the result given by refs[1,2].

Therefore, through direct thermodynamical calculations for two kinds of extreme 2D black holes, we come to a conclusion that the different results of entropy in fact come from two different treatments.

Now we are facing a puzzle. In statistical physics and thermodynamics, entropy is a function of state only, and does not depend on the history or the process of how the system arrives at the equilibrium state as well as the different treatment of mathematics. But now we use two different treatments for orders of limits to arrive at the final state, namely, the EBH state satisfying the same extreme condition, we get two different values of the entropy. The puzzle is similar to that in 4D RN cases and need further discussions.

### 3 Geometry and topology

In this section, we study the relation between the geometrical properties and two different treatments of taking different limits in detail.

Consider the Euclidean metric of 2D CDBH

$$ds^2 = g(r)d\tau^2 + g(r)^{-1}dr^2 \quad (31)$$

where  $g(r)$  has the form of Eq.(3). Taking the new variable  $\tau_1 = 2\pi T_c \tau$  where  $0 \leq \tau_1 \leq 2\pi$ , then Eq(31) becomes

$$ds^2 = (\beta/2\pi)^2 d\tau_1^2 + dl^2 \quad (32)$$

$\beta$  is the inverse local temperature and  $l$  is the proper distance. The equilibrium condition of the spacetime reads

$$\beta = \beta_0[g(r_B)]^{1/2}, 1/\beta_0 = T_c = g'(r_+)/4\pi \quad (33)$$

Let us choose the coordinate according to

$$r - r_+ = 4\pi T_c b^{-1} \sinh^2(x/2), b = g'(r_+)/2 \quad (34)$$

For the treatment (A), taking the limit  $r_+ \rightarrow r_B$  first, where the hole tends to occupy the entire cavity, the region  $r_+ \leq r \leq r_B$  shrinks and we can expand  $g(r)$  in the power of series  $g(r) = 4\pi T_c(r - r_+) + b(r - r_+)^2 + \dots$  near  $r = r_+$ . After substitution into Eqs.(32,33), and take the extreme limit  $r_+ = r_- = r_B$ ,  $b = \lambda^2$ , Eq(31) can be expressed as

$$ds^2(A) = \lambda^{-2}(\sinh^2 x d\tau_1^2 + dx^2) \quad (35)$$

This is just the 2D counterpart of the Bertotti-Robinson(BR) spacetime [27]. However, for the treatment (B), we start from the original extreme 2D CDBH,  $g(r) = (1 - me^{-\lambda r})^2$ . Introducing the variable  $r - r_+ = r_B \rho^{-1}$  and expand the metric coefficients near  $r = r_+$ , we obtain

$$ds^2(B) = \rho^{-2}(\lambda^2 r_B^2 d\tau^2 + d\rho^2/\lambda^2) \quad (36)$$

Now we are in a position to discuss the properties of Eqs(35,36). The horizons of the EBH got in the treatment (A) is determined by

$$g = 1/4g'(r_+)b^{-1} \sinh^2 x = 0 \quad (37)$$

So the horizon locates at finite  $x$ , say  $x = 0$ . The proper distance between the horizon and any other point is finite. However for the original extreme 2D CDBH (36), the horizon is detected by

$$g = \lambda^2 r_B^2 \rho^{-2} = 0 \quad (38)$$

therefore, the horizon is at  $\rho = \infty$ . The distance between the horizon and any other  $\rho < \infty$  is infinite. It is this difference here that gives rise to the qualitatively different topological features of the EBHs.

To exhibit these different topological features, we calculate the Euler characteristic of these two EBHs directly. The formula for the calculation of the Euler characteristic in 2D cases is [28]

$$\chi = \frac{1}{2\pi} \int R_{1212} e^1 \wedge e^2 \quad (39)$$

For the nonextreme 2D CDBH, adopting its metric, and subtracting the asymptotically flat space's influence [3,28], we arrive at

$$\chi = -\frac{1}{2\pi} \beta_0 [-m\lambda e^{-\lambda r} + q^2 \lambda e^{-2\lambda r}]_{r_+} = 1 \quad (40)$$

This result is in accordance with that of the multi-black-holes obtained in [3].

It is easy to extend the calculation of  $\chi$  to the cases of EBHs. For the EBH developed from the treatment (A), the Euler characteristic can be directly got by taking  $r_+ = r_B$  first and  $m = q$  afterwards,

$$\begin{aligned} \chi(A) &= -\frac{1}{2\pi} \beta_0 [-m\lambda e^{-\lambda r} + q^2 \lambda e^{-2\lambda r}]_{r_+=r_B} |_{extr} \\ &= \frac{2\pi(m + \sqrt{m^2 - q^2})\lambda\sqrt{m^2 - q^2}}{2\pi\lambda\sqrt{m^2 - q^2}(m + \sqrt{m^2 - q^2})} |_{extr} = 1 \end{aligned} \quad (41)$$

The same as that of the NEBH. However for the original extreme 2D CDBH, using the limit procedure (B) and from Eq.(40), we have

$$\chi(B) = -\frac{1}{2\pi} \beta_0 [-m\lambda e^{-\lambda r} + m^2 \lambda e^{-2\lambda r}]_{r_+} \quad (42)$$

The horizon for the original EBH is  $r_+ = \frac{1}{\lambda} \ln m$ , so

$$\chi(B) = 0 \quad (43)$$

It is quite different from that of the NEBH.

In order to obtain the property in general, we precede our discussion to 2D Lowe-Strominger black hole. The Euclidean metric has the same form as Eq.(31), but now

$$g(r) = -m + \lambda^2 r^2 + \frac{J^2}{4r^2} = \frac{\lambda^2}{r^2} (r^2 - r_+^2)(r^2 - r_-^2) \quad (44)$$

Using the same treatment (A) as that of 2D CDBH, namely, taking the boundary condition first and then the extreme limit, we find that the metric can be expressed as

$$ds^2(A) = (4\lambda^2)^{-1} (d\tau_1^2 \sinh^2 x + dx^2) \quad (45)$$

It is a 2D counterpart of BR spacetime again. While starting from the original EBH,  $g(r) = \frac{\lambda^2}{r^2}(r^2 - r_+^2)^2$ , as the original extreme 2D CDBH, the metric can be written as

$$ds^2(B) = \rho^{-2}(4\lambda^2 r_B^2 d\tau^2 + \frac{1}{4\lambda^2} d\rho^2) \quad (46)$$

The location of the horizon can be found directly for these two expressions of metric. For Eq.(45), from  $g = \frac{1}{4}g'(r_+)b^{-1}\sinh^2 x = 0$ , the horizon is at finite  $x$ , say  $x = 0$ . While for Eq.(46), the horizon is determined by  $g = 4\lambda^2 r_B \rho^{-2} = 0$ , so it is at  $\rho = \infty$ . The proper distances between a horizon and any other point are finite and infinite for Eqs.(45,46) respectively.

Applying Eq(39), we can also get the results for the Euler characteristic for these two kinds of 2D Lowe-Strominger black holes. Substituting the metric in Eq(39), the Euler characteristic for the NEBH reads

$$\chi = -\frac{\beta_0}{2\pi}[-\lambda^2 r + \frac{J}{4r^3}]_{r_+} = 1 \quad (47)$$

where we have used Eq(17) and (20). The Euler characteristic for the EBHs are obvious. For the EBH got in the treatment (A), we obtain

$$\chi(A) = -\frac{\beta_0}{2\pi}[-\lambda^2 r + \frac{J^2}{4r^3}]_{r_+=r_B|_{extr}} = 1 \quad (48)$$

However, in treatment (B), we find

$$\chi(B) = -\frac{\beta_0}{2\pi\lambda^2}[-\lambda^4 r + \frac{m^2}{4r^3}]_{r_+} \quad (49)$$

Considering  $r_+^2 = \frac{m}{2\lambda^2}$  for the original EBH, we finally get

$$\chi(B) = 0 \quad (50)$$

These results clearly show that the different treatments result in different geometrical and topological properties. The direct relation to the priority of taking limits, rather than introducing additional condition[3], makes the outcomes more concise and explicit. Different topological results obtained here make us easier to accept the different classical entropy derived for 2D EBHs. We find that in addition to 4D cases claimed in [3,4,12], in the 2D cases, the topology and the EBHs' classical entropy are closely related.

## 4 Quantum entropy

An early suggestion by 't Hooft[18] was that the fields propagating in the region just outside the horizon give the main contribution to the black hole entropy. The entropy of the black hole system arises from

entanglement[21-23]. Many methods, for example, the brick wall model[12,13], Pauli-Villars regular theory[20],etc., have been suggested to calculate the quantum effects of entropy in WKB approximation or in the one-loop approximation. Let's first study the 2D CDBH and suppose the CDBH is enveloped by a scalar field, and the whole system, the hole and the scalar field, are filling in a cavity. The wave equation of the scalar field is

$$\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi) - M^2\phi = 0 \quad (51)$$

Substituting the metric Eq.(2) into Eq.(51), we find

$$E^2(1 - 2me^{-\lambda r} + q^2e^{-2\lambda r})^{-1}f + \frac{\partial}{\partial r}[(1 - 2me^{-\lambda r} + q^2e^{-2\lambda r})\frac{\partial f}{\partial r}] - M^2f = 0 \quad (52)$$

Introducing the brick wall boundary condition[18]

$$\begin{aligned} \phi(x) &= 0 \text{ at } r = r_+ + \epsilon \\ \phi(x) &= 0 \text{ at } r = L \end{aligned}$$

and calculating the wave number  $K(r, E)$  and the free energy  $F$ , we get

$$K^2(r, E) = (1 - 2me^{-\lambda r} + q^2e^{-2\lambda r})^{-1}[(1 - 2me^{-\lambda r} + q^2e^{-2\lambda r})^{-1}E^2 - M^2] \quad (53)$$

$$F_{QM} = \frac{\pi}{6\beta^2\lambda}[\frac{1}{2}\ln(R^2 - 2mR + q^2) + \frac{m}{2\sqrt{m^2 - q^2}}\ln\frac{R - m - \sqrt{m^2 - q^2}}{R - m + \sqrt{m^2 - q^2}}] \quad (54)$$

where  $R = e^{\lambda(r_+ + \epsilon)}$ , and  $\epsilon \rightarrow 0$  is the coordinate cutoff parameter. To extend the above discussion to EBH, we are facing two limits  $\epsilon \rightarrow 0$  and  $q \rightarrow m$  again. It can be proved that Eq(54) depends on the order of taking these two limits. We find for treatment (A) which we first take boundary limit  $\epsilon \rightarrow 0$  and then the extreme limit  $q \rightarrow m$

$$F_{QM}(A) = -\frac{\pi}{6\beta^2\lambda}\ln(\frac{1}{m\lambda\epsilon} + \frac{m}{2\sqrt{m^2 - q^2}}\ln\frac{2\sqrt{m^2 - q^2}}{\lambda\epsilon(m + \sqrt{m^2 - q^2})}) \quad (55)$$

We just leave the term  $\sqrt{m^2 - q^2}$  in the second term of Eq.(55) for the moment for the following discussions.

But for treatment (B) by first adopt the extreme limit and then the boundary limit

$$F_{QM}(B) = -\frac{\pi}{6\beta^2\lambda}(\frac{m}{m\lambda\epsilon} + \ln\frac{1}{m\lambda\epsilon}) \quad (56)$$

Similar to the classical case, different expressions for free energy appear here due to different priority of taking different limits. Through the entropy formula  $S = \beta^2(\partial F/\partial \beta)$ , we obtain

$$S_{QM}(A) = \frac{\pi}{3\beta\lambda} \left( \ln \frac{1}{m\lambda\epsilon} + \frac{m}{2\sqrt{m^2 - q^2}} \ln \frac{2\sqrt{m^2 - q^2}}{\lambda\epsilon(m + \sqrt{m^2 - q^2})} \right) \quad (57)$$

$$S_{QM}(B) = \frac{\pi}{3\beta\lambda} \left( \frac{m}{m\lambda\epsilon} + \ln \frac{1}{m\lambda\epsilon} \right) \quad (58)$$

We conclude that the entropy on the black hole background also depends on two different treatments.

Now we turn to study 2D Lowe-Strominger model. We suppose the 2D black hole is enveloped by a scalar field, and the whole system are filling in a cavity. Substituting the metric Eq(17) into Eq(51), we get the radial equation as

$$E^2(-m + \lambda^2 r^2 + \frac{J^2}{4r^2})^{-1} f + \frac{\partial}{\partial r} [(-m + \lambda^2 r^2 + \frac{J^2}{4r^2}) \frac{\partial f}{\partial r}] - M^2 f = 0 \quad (59)$$

The wave number is

$$K^2 = (-m + \lambda^2 r^2 + \frac{J^2}{4r^2})^{-1} [(-m + \lambda^2 r^2 + \frac{J^2}{4r^2})^{-1} E^2 - M^2] \quad (60)$$

and the semiclassical quantization condition is

$$n\pi = \int_{r_+ + \epsilon}^L dr K(r, E) \quad (61)$$

The free energy satisfies

$$\begin{aligned} \beta F &= \sum_n \log(1 - e^{-\beta E}) \\ &= \int dn \log(1 - e^{-\beta E}) \\ &= -\frac{\beta}{\pi} \int dE (e^{\beta E} - 1)^{-1} f(r) \end{aligned} \quad (62)$$

where

$$f(r) = \int_{r_+ + \epsilon}^L dr (-m + \lambda^2 r^2 + \frac{J^2}{4r^2})^{-1} \sqrt{E^2 - M^2(-m + \lambda^2 r^2 + \frac{J^2}{4r^2})} \quad (63)$$

Expanding to the powers of  $M$  in the limit  $\epsilon \rightarrow 0$ , the leading contribution can be obtained as

$$f(r)_{r=r_+ + \epsilon} = -\frac{1}{\lambda^2} \left( \frac{A}{2r_+} \ln \frac{\epsilon}{2r_+ + \epsilon} + \frac{B}{2r_-} \ln \frac{r_+ - r_- + \epsilon}{r_+ + r_- + \epsilon} \right) \quad (64)$$

where

$$A = \frac{r_+^2}{r_+^2 - r_-^2}, \quad B = -\frac{r_-^2}{r_+^2 - r_-^2} \quad (65)$$

For treatment (A),

$$F(A) = -\frac{\pi}{6\beta^2} \left( \frac{1}{4r_+\lambda^2} \ln \frac{2r_+}{\epsilon} + \frac{r_-}{2\lambda^2(r_+^2 - r_-^2)} \ln \frac{2r_+(r_+ - r_-)}{\epsilon(r_+ - r_-)} \right) \quad (66)$$

As in Eq.(55), we leave  $r_+ - r_-$  for the moment.

For treatment (B), the free energy is

$$F(B) = -\frac{\pi}{6\beta^2} \left( \frac{1}{4\lambda^2\epsilon} + \frac{1}{4r_+\lambda^2} \ln \frac{2r_+}{\epsilon} \right) \quad (67)$$

Using the thermodynamic formula we find

$$S_{QM}(A) = \frac{\pi}{3\beta\lambda^2} \left( \frac{1}{4r_+} \ln \frac{2r_+}{\epsilon} + \frac{r_-}{2(r_+^2 - r_-^2)} \ln \frac{2r_+(r_+ - r_-)}{\epsilon(r_+ - r_-)} \right) \quad (68)$$

$$S_{QM}(B) = \frac{\pi}{3\beta\lambda^2} \left( \frac{1}{4\epsilon} + \frac{1}{4r_+} \ln \frac{2r_+}{\epsilon} \right) \quad (69)$$

respectively.

We find that two different values of entropies of scalar field on the backgrounds of extreme holes exhibit again. The results of 2D CDBH and Lowe-Strominger black holes tell us that the entropy of the scalar field on the EBHs' background depend on the limits procedures as well.

## 5 Conclusions and discussions

Through direct thermodynamic calculations, we have shown that corresponding to two different treatments of the orders of taking the boundary limit and black hole extreme limit, the classical entropies of 2D extreme CDBH and Lowe-Strominger black hole have different values, namely, zero and Bekenstein-Hawking value. We have shown that the geometrical and topological properties are also dependent on these two different treatments by direct provement. Since the extreme conditions are all satisfied for the discussed EBHs, these profoundly different geometrical and topological properties lead us to an impression that there are two kinds of EBHs which are classified by different topologies. One is the original EBH as Hawking et al. claimed and due to peculiar topology of this EBH, this kind of EBH cannot be formed from their nonextreme counterpart and can only be prepared in the early universe. While the other EBH obtained by treatment (A) with the same topology as that of NEBH can be developed from their NEBH counterpart. Entropies and topological properties for EBHs studied in 4D cases[3,4,12] as well as in 2D cases in our paper suggest that there are close relation between the topology and the classical entropy for EBHs. Our results have been supported recently by Pretorius, Vollick and Israel [31].

Using the brick wall model, we have shown that under WKB approximation the entropy of scalar field on the background of extreme 2D CDBH and Lowe-Strominger black hole have two values given by Eqs(57,58) and (68,69), respectively. These results support our argument from the quantum point of view that the backgrounds of EBHs are different due to different topologies.

Besides the usual ultraviolet divergence  $\epsilon \rightarrow 0$ , which has been found for both 4D and 2D NEBH cases [18-23] and was suggested capable of being overcome by different renormalization methods [29,30], a new divergent term for Zaslavskii's treatment emerges in Eqs.(57,68) and is absent in Hawking's treatment. To understand this difference, let's go over the arguments of Hawking et al. and Zaslavskii again. Owing to different topological properties, Hawking et al. claimed that their EBH and its NEBH counterpart are completely different objects and their EBH is the original EBH, which can only be prepared at the beginning of the universe. In Hawking's treatment, since we put  $m = q$  and  $r_+ = r_-$  first for 2D CDBH and Lowe-Strominger black hole respectively, and then calculate their entropies by using usual thermodynamical approaches [24], naturally their quantum entropy includes ultraviolet divergence only. But for the EBH obtained by using Zaslavskii's treatment, it can be developed from the nonextreme counterpart by taking the extreme limits, therefore new divergent terms appear here. Keeping in mind that the entropy of scalar fields in the EBH background was derived by WKB approximation, the divergent quantum entropy due to taking extreme limits reflects in fact the divergent quantum fluctuations of the entanglement entropy of the whole system including the EBH and the scalar field. In statistical physics we know that infinite fluctuation breaks down the rigorous meanings of thermodynamical quantities and is just the characteristic of the point of phase transition. This conclusion is in good agreement with the previous studies about the phase transition of black holes by using Landau-Lifshitz theory [32-34]. So a phase transition will happen when a NEBH approaches to the EBH with nonextreme topology obtained by using Zaslavskii treatment. The new divergent terms appears in the quantum entropy here gives the quantum understanding of phase transition and supports previous classical arguments.

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## References

- [1] S. W. Hawking, G. Horowitz and S. Ross, Phys. Rev. D 51, 4302 (1995)

- [2] C. Teitelboim, Phys. Rev. D 51, 4315 (1995)
- [3] G. W. Gibbons and R. E. Kallosh, Phys. Rev. D 51, 2839 (1995)
- [4] S. Liberati and G. Pollofrone, Phys. Rev. D 56, 6458 (1997)
- [5] O. B. Zaslavskii, Phys. Rev. Lett. 76, 2211 (1996)
- [6] J. M. Maldacena and A. Strominger, Phys. Rev. Lett. 77, 428 (1996)
- [7] O. B. Zaslavskii, Phys. Rev. D 56, 2188 (1997)
- [8] O. B. Zaslavskii, Phys. Rev. D 56, 6695 (1997)
- [9] A. Ghosh and P. Mitra, Phys. Rev. Lett. 78, 1858 (1997)
- [10] O. B. Zaslavskii, Phys. Rev. Lett. 80, 3412 (1998)
- [11] O. B. Zaslavskii, hep-th/9804090
- [12] B. Wang, R. K. Su and P. K. N. Yu, Phys. Rev. D (in press, AIP 100820prd)
- [13] B. Wang and R. K. Su, Phys. Lett. B432, 69 (1998)
- [14] B. Wang, R. K. Su and P. K. N. Yu, Phys. Lett. B 438, 47 (1998)
- [15] C. Nappi and A. Pasquinucci, Mod. Phys. Lett. A 7, 3337 (1992)
- [16] M. D. McGuigan, C. R. Nappi and S. A. Yost, Nucl. Phys. B 375, 121 (1992)
- [17] D. Lowe and A. Strominger, Phys. Rev. Lett 73, 1468 (1994)
- [18] G. 't Hooft, Nucl. Phys. B 256, 727 (1985)
- [19] A. Ghosh and P. Mitra, Phys. Rev. Lett. 73, 2521 (1994); Phys. Lett. B 357, 295 (1995)
- [20] J. G. Demers, R. Lafrance and R. C. Myers, Phys. Rev. D 52, 2245 (1995)
- [21] S. N. Solodukhin, Phys. Rev. D 54, 3900, (1996), *ibid.* D 52, 7046 (1995)  
M. Srednicki, Phys. Rev. Lett 71, 666 (1993)  
V. Frolov and I. Novilov, Phys. Rev. D 48, 4545 (1993)
- [22] C. Callen and F. Wilczek, Phys. Lett. B 333, 55 (1994)

- [23] J. Ellis, N. E. Mavromato, E. Winstanley, and D. V. Nanopoulos, *Mod. Phys. Lett. A* 12, 243 (1997)
- [24] H. W. Braden, J. D. Brown, B. F. Whiting and J. W. York, *Phys. Rev. D* 42, 3376 (1990)
- [25] A. Kumar and K. Ray, *Phys. Rev. D* 51, 5954 (1995); *Phys. Lett. B* 351, 431 (1995)
- [26] R. Kallosh, T. Ortin and A. Peet, *Phys. Rev. D* 47, 5400 (1993)
- [27] I. Robinson, *Bull. Acad. Pol. Sci.* 7, 351 (1959)  
B. Bertotti, *Phys. Rev.* 116, 1331 (1959)
- [28] T. Eguchi, P. B. Gikney and A. J. Hanson, *Phys. Rep.* 66, 6 (1980)
- [29] L. Susskind and J. Uglum, *Phys. Rev. D* 50, 2700 (1994)
- [30] S. N. Solodukhin, *Phys. Rev.* 51, 609, 618, (1995)
- [31] F. Pretorius, D. Vollick and W. Israel, *Phys. Rev. D* 57, 6311 (1998)
- [32] R. K. Su, R. G. Cai, P. K. N. Yu, *Phys. Rev. D* 50, 2932 (1994); R. G. Cai, R. K. Su, P. K. N. Yu,  
*Phys. Rev. D* 48, 3473 (1993); *D* 52, 6186 (1995)
- [33] O. Kaburaki, *Phys. Lett. A* 217, 316 (1996)
- [34] C. O. Lousto, *Phys. Rev. D* 51, 1733 (1995)